# GENERALIZED ABSOLUTELY MONOTONE FUNCTIONS

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ABSTRACT

The concept of absolutely monotone functions is generalized by replacing the conditions $\phi^{(k)}(t) \geq 0, k=0, 1, \ldots$ by an infinite sequence of differential inequalities $\phi(t) \geqq 0, L_k\phi(t) \geqq 0, k = 1, 2, \ldots$, where the $L_k$ are differential operators of a special type. It is shown that these functions have a valid series expansion in terms of basic functions associated with the operators $L_k$.

A function $\phi(t)$ defined on $(a,b)$ which satisfies $\phi^{(k)}(t) \geqq 0$ for $t \in (a,b)$ and all $k = 0,1,2,\cdots$ is called absolutely monotone. For a detailed discussion of the history and applications of the notion of absolute monotonicity with reference to areas of classical mathematics see [1, Chapter 4]. A multivariate generalization of the concept of absolute monotonicity is discussed in Bochner [5, Chapter 4].

It is a familiar fact that an absolutely monotone function can be expanded in a power series

$$\phi(t) = \sum_{k=0}^{\infty} \phi^{(k)}(0+) \frac{(t-a)^k}{k!} \tag{1}$$

convergent for $|t-a| < b-a$. The purpose of this note is to generalize the concept of absolute monotonicity and to establish the analogue of (1).

Let $\{w_i(t)\}_{i=0}^{\infty}$ be an infinite sequence of positive functions, each of class $C^{\infty}[a,b]$. With these functions we associate the sequence of first order differential operators

$$D_i f(t) = \frac{d}{dt} \frac{1}{w_i(t)} f(t), \quad i = 0,1,2,\cdots \tag{2}$$

DEFINITION 1. A function $\phi(t)$ defined on $(a,b)$ is called a "generalized absolutely monotone" (abbreviated G.A.M.) with respect to $\{w_i\}_{i=0}^{\infty}$ provided $\phi(t)$ is of class $C^{\infty}$ on the open interval $(a,b)$ and satisfies the inequalities

$$\phi(t) \geqq 0 \text{ and } (D_k D_{k-1} \cdots D_0)\phi(t) \geqq 0 \tag{3}$$

for all $t \in (a,b)$, $k = 0,1,\cdots$.

The special choice $w_i(t) \equiv 1$, $i = 0,1,\cdots$ corresponds to the standard notion of

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absolute monotonicity. A few remarks on other generalizations are in order. A homogeneous version of (3) was partly investigated by Hirschman and Widder [2]. Their case corresponds to the circumstance where the differential operator $L_k = D_k D_{k-1} \cdots D_0$ reduces to a linear $k+1^{st}$ order differential operator with constant coefficients. The operator $L_k$ for general $\{w_i(t)\}$ induces a linear $k+1^{st}$ order differential operator with variable coefficients admitting a factorization into linear terms, i.e., $L_k = (D+\lambda_1(x))(D+\lambda_2(x))\cdots(D+\lambda_{k+1}(x))$ (see Karlin and Studden [3]). Differential operators of this type were first singled out by Pólya [4] in the course of developing certain generalizations of the mean value theorem.

Turning to the task at hand we will first describe a geometric characterization of the class of G.A.M. functions involving certain convex cones. To this end we introduce the special functions

$$u_k(t) = w_0(t)\int_a^t w_1(\xi_1)\int_a^{\xi_1} w_2(\xi_2)\cdots\int_a^{\xi_{k-1}} w_k(\xi_k)\,d\xi_k\cdots d\xi_1, \qquad k = 0,1,\cdots, t\in[a,b]. \tag{4}$$

It is straightforward to check that $u_k(t)$ is the unique solution of the $k+1^{st}$ order differential equation

$$L_k u = (D_k D_{k-1}\cdots D_0)u = 0 \tag{5}$$

subject to the initial conditions $u^{(i)}(a) = \delta_{ki}$ $i = 0,\cdots, k^{-1}$ and $u^{(k)}(a) = \prod_{i=0}^k w_i(a)$. Notice that in the special case $w_i(t) = 1$, we have $u_k(t) = t^k/k!$

The functions $u_0, u_1,\cdots,u_n$ constitute an extended Tchebycheff system on $(a,b)$, i.e., any non-trivial linear combination $\sum_{i=0}^n c_i u_i(t)$ with real coefficients can vanish at most $n$ times counting multiplicities. We refer the reader to the book by Karlin and Studden [3] which contains an elaborate study of the theory of Tchebycheff system with emphasis on a geometric point of view.

With respect to $\{u_0,\cdots,u_n\}$ we generate a convex cone $\mathscr{C}(u_0,\cdots,u_n)$ of functions in the following manner.

Definition 2. A function $\psi(x)$ belongs to $\mathscr{C}(u_0,\cdots,u_n)$ (and is called "convex with respect to $(u_0,\cdots,u_n)$") if for every set of points $\{x_i\}_{i=1}^{n+2}$ satisfying

$$a < x_1 < x_2 < \cdots < x_{n+2} < b$$

the determinant inequality

$$\begin{vmatrix} u_0(x_1), & \cdots & u_0(x_{n+2}) \\ u_1(x_1), & \cdots & u_1(x_{n+2}) \\ \vdots & & \vdots \\ u_n(x_1), & \cdots & u_n(x_{n+2}) \\ \psi(x_1), & \cdots & \psi(x_{n+2}) \end{vmatrix} \geqq 0 \tag{6}$$

prevails.

For functions $\psi(x)$ which are $n+1$ times continuously differentiable, it is proved in [3] that $\psi$ belongs to $\mathscr{C}(u_0,\cdots,u_n)$ if

$$(D_nD_{n-1}\cdots D_0)\psi(x)\geqq 0 \qquad x\in(a,b). \tag{7}$$

The converse is also valid provided the differential operator in (7) is suitably interpreted (see Karlin and Studden (3) chap. 11 and Ziegler [6] for further details).

Consider now the intersection cone

$$\mathscr{C}_A=\mathscr{C}^+\cap\left[\bigcap_{n=0}^{\infty}\mathscr{C}(u_0,u_1,\cdots,u_n)\right] \tag{8}$$

where $\mathscr{C}^+$ denotes the cone of continuous non-negative functions defined on $(a,b)$. It is proved in [3] that $\phi$ belongs to $\mathscr{C}_A$ if and only if $\phi$ is infinitely continuously differentiable, $\phi(x)\geqq 0$ on $(a,b)$ and (7) holds for $n=0,1,\cdots$. Thus, the convex cone $\mathscr{C}_A$ coincides with the class of G.A.M. functions.

We quote the following Taylor-type formula needed later.

Let $f(x)$ be any $n+1$ times continuously differentiable function defined on $(a,b)$ such that $\lim_{x\to a^+}[d^nf(x)]/dx^n$ exists for each $n$(*). Then

$$f(x) = \int_a^b \phi_n(x;t)L_nf(t)\,dt + \sum_{k=0}^{n} \rho_k(a^+)u_k(x) \tag{9}$$

where

$$\rho_0(a^+) = \frac{f(a^+)}{w_0(a)} \quad \rho_k(a^+) = \frac{D_{k-1}\cdots D_0f(a^+)}{w_k(a)}, \quad k=1,2,\cdots$$

and

$$\phi_n(x;t) = \begin{cases} 0 & a\leqq x<t \\ w_0(x)\int_t^x w_1(\xi_1)\int_t^{\xi_1} w_2(\xi_2)\cdots\int_t^{\xi_{n-1}} w_n(\xi_n)\,d\xi_n\cdots d\xi_1 & x\leqq t\leqq b. \end{cases} \tag{10}$$

The validation of (9) appears in [3], see also [6].

We are now prepared to state the principal theorem of the paper.

THEOREM 1. *Let $\{w_i\}_0^\infty$ be a sequence of positive $C^\infty$ functions defined on $[a,b]$ and let $\{u_k\}_0^\infty$ be the ECT-system associated with the $\{w_i\}$ as in* (4). *Let $m_i(x;y)$ and $M_i(x;y)$ be defined by*

$$0<m_i(x;y)=\min_{x\leqq t\leqq y} w_i(t)\leqq \max_{x\leqq t\leqq y} w_i(t)=M_i(x;y),\ i=0,1,\cdots \tag{11}$$

* Observe that (7) together with $\phi(x)\geqq 0$ implies that $\lim_{x\to a^+}[d^n\phi(x)]/dx^n$ exists and is finite for all $n=0,1,\ldots$ Therefore the formula (9) is applicable whenever $f(x)$ is G.A.M.

*If for every* $c \in [a,b]$ *there exists a* $d$, $c < d < b$ *and an* $\varepsilon > 0$ *such that*

$$\lim_{n \to \infty} \left( \prod_{i=0}^{n} \frac{M_i(c;d)}{m_i(c;d)} \right) \varepsilon^n = 0 \tag{12}$$

*then each* $\phi \in \mathscr{C}^+ \cap [\bigcap_{k=0}^{\infty} \mathscr{C}(u_0, \cdots, u_k)]$ *(i.e.,* $\phi$ *is a G.A.M. function) possesses a representation*

$$\phi(t) = \sum_{k=0}^{\infty} \rho_k(a^+) u_k(t), \qquad t \in (a,b) \tag{13}$$

*where*

$$\rho_0(t) = \frac{\phi(t)}{w_0(t)}, \quad \rho_k(t) = \rho_k(t;\phi) = \frac{D_{k-1} \cdots D_0 \phi(t)}{w_k(t)}, \quad k = 1, 2, \cdots$$

REMARK 1. The requirement (12) is manifestly fulfilled if $w_i(t)$ are uniformly bounded from above and below.

REMARK 2. The convergence in (13) is uniform on every compact subinterval of $(a,b)$. This follows immediately from Dini's theorem concerning monotone convergence of functions owing to the fact that all the terms including the limit function are non-negative and continuous. Furthermore, if $\phi(t)$ is continuous at the end point $b$ then (13) is valid also at $b$. To prove this statement we observe, since $\phi(t)/w_0(t)$ is continuous at $b$ and $\phi(t)/w_0(t)$ is non-decreasing, that for prescribed $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ and so small that

$$\frac{\phi(b)}{w_0(b)} \leqq \frac{\phi(b-\eta)}{w_0(b-\eta)} + \varepsilon .$$

Now the convergence of (13) at $b - \eta$ implies that for $n$ large enough

$$\begin{aligned} \frac{\phi(b-\eta)}{w_0(b-\eta)} &\leqq \sum_{k=0}^{n} \rho_k(a^+) \frac{u_k(b-\eta)}{w_0(b-\eta)} + \varepsilon \\ &\leqq \sum_{k=0}^{n} \rho_k(a^+) \frac{u_k(b)}{w_0(b)} + \varepsilon \\ &\leqq \sum_{k=0}^{\infty} \rho_k(a^+) \frac{u_k(b)}{w_0(b)} + \varepsilon , \end{aligned}$$

the second inequality resulting since $u_k(t)/w_0(t)$ is non-decreasing. Letting $\varepsilon \downarrow 0$, we see that

$$\frac{\phi(b)}{w_0(b)} \leqq \sum_{k=0}^{\infty} \rho_k(a^+) \frac{u_k(b)}{w_0(b)} .$$

On the other hand, since $\phi(t)/w_0(t)$ is increasing, we obtain

$$\frac{\phi(b)}{w_0(b)} \geqq \frac{\phi(b-\varepsilon)}{w_0(b-\varepsilon)} \geqq \sum_{k=0}^{n} \rho_k(a^+) \frac{u_k(b-\varepsilon)}{w_0(b-\varepsilon)} .$$

It follows by letting $\varepsilon \downarrow 0$, that

$$\frac{\phi(b)}{w_0(b)} \geqq \sum_{k=0}^{n} \rho_k(a^+) \frac{u_k(b)}{w_0(b)}$$

and this inequality holds for all $n$. Therefore

$$\phi(b) = \sum_{k=0}^{\infty} \rho_k(a^+) u_k(b).$$

For $t = a$ the situation is much simpler since $u_k(a) = 0$, $k = 1, 2, \cdots$

Thus if $\phi$ is continuous at $t = a$ and $t = b$ then the convergence in (13) is uniform over $[a, b]$.

**Proof of Theorem 1.** Let $\phi$ belong to $\mathscr{C}^+ \cap [\bigcap_{n=0}^{\infty} \mathscr{C}(u_0, \cdots, u_n)]$. Then the functions $\rho_n(t)$, $n=0,1,\cdots$, are non-negative, continuous and non-decreasing on $(a, b)$. Thus, $\rho_n(a^+)$, $n = 0, 1, \cdots$ exist, are non-negative and the generalized Taylor formula (9) applies.

Since

$$s_n(t) = \sum_{k=0}^{n} \rho_k(a^+) u_k(t)$$

is a non-decreasing sequence bounded above by $\phi(t)$ we may infer that $s_n(t)$ converges to $s(t) < \infty$. We define

$$g(t) = \phi(t) - s(t) = \lim_{n \to \infty} \int_a^b \phi_n(t; x) \, d\rho_n(x) \tag{14}$$

and it is required to prove that $g(t) \equiv 0$ for $t \in [a, b)$. For this purpose the following lemma is useful

LEMMA. *Suppose for each* $\phi \in \mathscr{C}_A = \mathscr{C}^+ \cap [\bigcap_{n=0}^{\infty} \mathscr{C}(u_0, \cdots, u_n)]$ and each $c \in [a, b)$, the *relation*

$$\lim_{n \to \infty} \int_c^b \phi_n(t; x) \, d\rho_n(x; \phi) = 0 \tag{15}$$

*is fulfilled for t in some non-degenerate interval* $[c, c + \varepsilon)$ *for* $\varepsilon > 0$ *which may depend on* $\phi$. *Then* $g(t) \equiv 0$ *for* $t \in [a, b)$.

**Proof.** By a result proved in [6],

$$\phi_n(t; x) \in \mathscr{C}^+ \cap [\bigcap_{k=0}^{m} \mathscr{C}(u_0, \cdots, u_k)], \qquad \text{for } n \geqq m$$

and therefore

$$g(t) = \lim_{n \to \infty} \int_a^b \phi_n(t; x) \, d\rho_n(x) \in \mathscr{C}^+ \cap [\bigcap_{k=0}^{m} \mathscr{C}(u_0, \cdots, u_k)].$$

This holds independently of $m$, and therefore $g(t) \in \mathscr{C}_A$.

As pointed out previously every member of $\mathscr{C}_A$ is automatically of class $C^\infty(a,b)$ and moreover $\rho_n(a^+;g)$ exists for all $n$. Now suppose to the contrary that $g(t) \not\equiv 0$ on $[a,b)$; then there exists a maximal interval connected to $a$ on which $g(t)=0$. We denote this interval by $[a,c^*]$, $c^* < b$ and $c^*$ exceeds $a$ by virtue of the hypothesis of the lemma. Since $g(t) \equiv 0$ for $t \in [a,c^*]$ and $g(t) \in C^\infty(a,b)$, the representation formula (9) applied to $g(t)$ (with respect to the interval $(c^*,b)$) reduces to

$$g(t) = \int_{c^*}^{b} \phi_n(t;x)\, d\rho_n(x;g).$$

Invoking the assertion of the lemma for $g(t)$, we infer that

$$g(t) = \lim_{n\to\infty} \int_{c^*}^{b} \phi_n(t;x)\, d\rho_n(x;g) = 0 \qquad t \in [c^*, c^* + \delta) \tag{16}$$

for some $\delta > 0$. This conclusion is in contradiction to the definition of $c^*$ and the proof of the lemma is complete.

We now prove that the hypothesis of the lemma is satisfied. Consulting (9), we see that for $a \leqq x \leqq t < d \leqq b$

$$\begin{aligned} \phi(d) &\geqq \int_x^d \phi_n(d;\xi)\, d\rho_n(\xi;\phi) \\ &= \int_x^d \phi_n(d;\xi)\, w_{n+1}(\xi)\, \rho_{n+1}(\xi)\, d\xi \\ &\geqq \rho_{n+1}(x) \int_x^d \phi_n(d;\xi)\, w_{n+1}(\xi)\, d\xi \\ &\geqq \rho_{n+1}(x) \int_t^d \phi_n(d;\xi)\, w_{n+1}(\xi)\, d\xi. \end{aligned}$$

Moreover, from the definition of $\phi_n(d;\xi)$

$$\int_t^d \phi_n(d;\xi)\, w_{n+1}(\xi)\, d\xi \geqq \left( \prod_{i=0}^{n+1} m_i(t;d) \right) \frac{(d-t)^{n+1}}{(n+1)!} \tag{17}$$

where

$$m_i(t;d) = \min_{t \leqq z \leqq d} [w_i(z),] \qquad i = 0, 1, \cdots, n+1.$$

Hence, if $a \leqq c \leqq t < d$, then

$$(18)\quad \begin{aligned}\int_c^b \phi_n(t;x)d\rho_n(x) &= \int_c^t \phi_n(t;x)w_{n+1}(x)\rho_{n+1}(x)dx \\ &\leqq \phi(d)\frac{(n+1)!}{(d-t)^{n+1}\prod_{i=0}^{n+1} m_i(t;d)}\int_c^t \phi_n(t;x)w_{n+1}(x)dx \\ &\leqq \phi(d)\left(\frac{t-c}{d-t}\right)^{n+1}\left(\prod_{i=0}^{n+1}\frac{M_i(c;t)}{m_i(t;d)}\right)\end{aligned}$$

where

$$M_i(c;t) = \max_{c \leqq z \leqq t} w_i(z), \qquad i = 0,1,\cdots,n+1 .$$

Using condition (12), it follows that

$$\lim_{n\to\infty}\int_c^b \phi_n(t;x)\,d\rho_n(x) = 0$$

for $t$ in some non-degenerate interval $[c, c+\delta)$. Q.E.D.

COROLLARY 1. *If $w_i(t)$, $i = 0,1,\cdots$ are uniformly bounded from above and below, then the expansion* (13) *holds.*

COROLLARY 2. *If $w_i(t)$, $i = 0,1,\cdots$ are non-decreasing functions of $t$, then* (13) *holds.*

Indeed, the estimate in (18) reduces to

$$(19)\qquad \int_c^b \phi_n(t;x)\,d\rho_n(x) \leqq \phi(b)\left(\frac{t-c}{d-t}\right)^{n+1}$$

since in this case $m_i(t;d) = M_i(c;t)$.

THEOREM 2. *If there exist two sequences $\{c_n\}_1^\infty$, $\{d_n\}_1^\infty$ and an integer $N_0$ such that*

a) *The functions $\phi_n(t;x)$ for $n \geqq N_0$ satisfy*

$$(20)\qquad c_n\frac{(t-x)^n}{n!} \leqq \phi_n(t;x) \leqq d_n\frac{(t-x)^n}{n!}, \qquad x \leqq t .$$

b) *There exists an $\varepsilon > 0$ such that*

$$(21)\qquad \lim_{n\to\infty}\frac{d_n}{c_n}\,\varepsilon^n \to 0\,;$$

*then the expansion* (13) *is valid.*

**Proof.** We observe first the formulae (see [7])

$$\int_t^\beta \phi_n(\beta;\xi)\, w_{n+1}(\xi)\, d\xi = \phi_{n+1}(\beta;t)$$

and

$$\int_c^t \phi_n(t;x)\, w_{n+1}(x)\, dx = \phi_{n+1}(t;c).$$

Proceeding as in the proof of Theorem 1, and replacing the estimates in (17) and (18) by the estimate in (20), we obtain

$$\int_a^b \phi_n(t;x)\, d\rho_n(x) \leqq \phi(\beta) \frac{\phi_{n+1}(t;c)}{\phi_{n+1}(\beta;t)} \leqq \phi(\beta) \left( \frac{t-c}{\beta - t} \right)^{n+1} \frac{d_{n+1}}{c_{n+1}}.$$

The validity of the theorem now follows by using (21). Q.E.D.

With the aid of the above theorem, we can now characterize the dual cone to $\mathscr{C}_A = \mathscr{C}^+ \cap [\bigcap_{n=0}^{\infty} \mathscr{C}(u_0, \cdots, u_n)]$.

DEFINITION 3. A signed measure $\mu$ of bounded variation on $(a, b)$ is said to belong to the dual cone $\mathscr{C}_A^*$ provided for every $\phi \in \mathscr{C}_A$ that at least one of the integrals $\int_a^b \phi\, d\mu_1$ or $\int_a^b \phi\, d\mu_2$ is finite (where $\mu = \mu_1 + \mu_2$ represents the Jordan decomposition of $\mu$) and

$$\int_a^b \phi\, d\mu \geqq 0.$$

THEOREM 3. *Let $\{w_i\}_0^\infty$ satisfy the requirements of Theorem 1. A signed measure $d\mu$ belongs to the dual of $\mathscr{C}_A$ if and only if*

$$\int_a^b u_i\, d\mu \geqq 0 \qquad i = 0, 1, \cdots \tag{22}$$

**Proof.** We know that $u_i \in \mathscr{C}_A$, $i = 0, 1, \cdots$ so that (22) is certainly necessary. The validity of (13) easily implies that the inequality (22) is also sufficient.

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